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# The Gyromagnetic Ratio of the Electron in a Coulomb Field

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We study the corrections to the gyromagnetic ratio $g$, of the electron, due to a Coulomb field. They are of order $\alpha^2$.

## 1. INTRODUCTION

It is well known that there is a change in the magnetic moment of an electron due to its motion in a Coulomb field (Breit, 1928; Margenau, 1940). Thus the gyromagnetic ratio $g$ of the electron also changes. Our purpose is to evaluate $g$ in the above case. We use the virial theorem (Vázquez, 1977). The variation of $g$ is of order $\alpha^2$, which is very small, but it has a suggestive interpretation. The interaction between the electron and the external Coulomb field is energetically weak, compared to the rest mass of the electron. But, in some way, the structure of the electron is modified by the external field. If, in a naive picture, we interpret the electron as a localized electromagnetic current, then we can imagine that this is modified by the field and thus there is a change in the magnetic moment of the electron.

Also we evaluate the relativistic corrections to the Zeeman effect (Bethe and Salpeter, 1957), in terms of the modifications of the gyromagnetic ratio.

## 2. HYDROGEN ATOM

The Dirac equation for the electron in a Coulomb field is

$$i\gamma^K \partial_K \psi - m\psi + \left(E + \frac{e^2}{r}\right)\gamma^0\psi = 0 \tag{2.1}$$

the system of units which we use is that in which $\hbar = c = 1$.



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Equation (2.1) may be separated in spherical coordinates (Bethe and Salpeter, 1957):

$$\psi = e^{-iEt}\begin{bmatrix} h(r) & Y_{jl_A}^M \\ if(r) & Y_{jl_B}^M \end{bmatrix} \tag{2.2}$$

Where $Y_{jl}^M$ are the spherical harmonics that satisfy the normalization condition

$$\int \left(Y_{jl}^M\right)^+ \left(Y_{jl}^M\right) d\Omega = 1 \tag{2.3}$$

And we have the radial equations

$$h' + \frac{1+\mathcal{K}}{r} h = \left(E + \frac{e^2}{r} + m\right) f$$

$$f' + \frac{1-\mathcal{K}}{r} f = -\left(E + \frac{e^2}{r} - m\right) h \tag{2.4}$$

with $\mathcal{K} = -(1+1)$ if $j = 1 - 1/2$, and $\mathcal{K} = 1$ if $j = 1 - 1/2$.

Applying the virial theorem (Vázquez, 1977) we obtain the integral condition

$$\int \left(m\bar{\psi}\psi - E\psi^+\psi\right) d^3x = 0 \tag{2.5a}$$

and in terms of the radial functions

$$\int_0^\infty f^2 r^2\, dr = \frac{m-E}{m+E} \int_0^\infty h^2 r^2\, dr \tag{2.5b}$$

this is a useful relation in order to get the gyromagnetic ratio.

The angular momentum associated to (2.2) is given by

$$S = M \int \psi^+ \psi\, d^3x = M \int_0^\infty (h^2 + f^2) r^2\, dr \tag{2.6}$$

and with the help of (2.5b) we get

$$S = \frac{2m}{m+E} M \int_0^\infty h^2 r^2\, dr \tag{2.7}$$

The magnetic moment is

$$\mathfrak{M}=\frac{1}{2}\int \mathbf{r}\times\mathbf{j}\,d^3x \tag{2.8}$$

where $j^K=e\bar{\psi}\gamma^K\psi$, with $e$ the electron charge. In order to compute $\mathfrak{M}$, it is useful to express (2.2) in the form

$$\psi=e^{-iEt}\begin{bmatrix} a_1e^{i(M-1/2)\varphi} \\ a_2e^{i(M+1/2)\varphi} \\ ia_3e^{i(M-1/2)\varphi} \\ ia_4e^{i(M+1/2)\varphi}\end{bmatrix} \tag{2.9}$$

where $a_i=a_i(r,\theta)$. Thus we get

$$\mathfrak{M}_1=\mathfrak{M}_2=0$$

$$\mathfrak{M}_3\equiv\mathfrak{M}=2\pi e\int_0^\infty\int_{-1}^{1}(a_1a_4-a_2a_3)r^3P_1^1d(\cos\theta)\,dr \tag{2.10}$$

We distinguish two cases:

(1) $j=l+1/2[\mathcal{K}=-(l+1)]$. Then

$$\mathfrak{M}=eM\frac{4(l+1)}{(2l+1)(2l+3)}\int_0^\infty fhr^3\,dr \tag{2.11}$$

and with the help of the radial equations (2.4) we get

$$\mathfrak{M}=\frac{-e}{2m}\frac{4M(l+1)}{(2l+1)(2l+3)}\frac{m+2E(l+1)}{m+E}\int_0^\infty h^2r^2\,dr \tag{2.12}$$

and the gyromagnetic ratio is

$$g=\frac{\mathfrak{M}}{[-(e/2m)S]}=2\frac{l+1}{(2l+1)(2l+3)}[1+2\varepsilon(l+1)] \tag{2.13}$$

where

$$\varepsilon=\frac{E}{M}=\left\{1+\left[\frac{\alpha}{n-\mathcal{K}+(\mathcal{K}^2-\alpha^2)^{1/2}}\right]^2\right\}^{-1/2}$$

Expanding $\varepsilon$ we get

$$g \simeq \frac{2(l+1)}{(2l+1)}\left(1-\frac{l+1}{2l+3}\frac{\alpha^2}{n^2}\cdots\right) \tag{2.14}$$

For the spherically symmetric states ($l=0$) we have

$$g=2-\tfrac{4}{3}(1-\varepsilon)\simeq 2-\tfrac{2}{3}\frac{\alpha^2}{n^2} \tag{2.15}$$

And if $\varepsilon \to 1$ (free state) then $g \to 2$.

(2)$j=l-1/2(\mathcal{K}=l)$. In this case

$$\mathfrak{M}=\frac{-4elM}{(2l+1)(2l-1)}\int_0^\infty fhr^3\,dr \tag{2.16}$$

and with the help of the radial equation (2.4) we get

$$g=2\frac{l}{(2l+1)(2l-1)}(2l\varepsilon-1) \tag{2.17}$$

$$g \simeq \frac{2l}{2l+1}\left(1-\frac{l}{2l-1}\frac{\alpha^2}{n^2}\cdots\right) \tag{2.18}$$

The first term in (2.14) and (2.18) is the Landé factor, which is independent on the energy.

*Remark 1.* If we consider the Gordon decomposition of the current

$$j^\mu=\frac{ie}{2m}\left[\bar{\psi}(\partial^\mu\psi)-(\partial^\mu\bar{\psi})\psi\right]-\frac{e^2}{m}A^\mu\bar{\psi}\psi-\frac{e}{2m}\partial_\nu(\bar{\psi}\partial^{\mu\nu}\psi) \tag{2.19}$$

we may evaluate the contributions to the magnetic moment due to the convection current ($\mathfrak{M}_c$) and the current associated with the intrinsic magnetization of the electron ($\mathfrak{M}_s$). In particular for the $S$ states we get

$$\mathfrak{M}_c=\frac{4\pi}{3}\frac{e}{m}\frac{\varepsilon-1}{1+\varepsilon}\int_0^\infty h^2r^2\,dr$$

$$\mathfrak{M}_s=\frac{4\pi}{3}\frac{e}{m}\frac{2+\varepsilon}{1+\varepsilon}\int_0^\infty h^2r^2\,dr \tag{2.20}$$

Thus we can define two gyromagnetic ratios $g_c$, $g_s$ $(g_c + g_s = g)$

$$g_s = 2\frac{2+\varepsilon}{1+2\varepsilon}$$

$$g_c = \frac{2}{3}\frac{4\varepsilon^2+\varepsilon-5}{1+2\varepsilon} \tag{2.21}$$

When $\varepsilon \to 1 \Rightarrow g_s \to 2$ and $g_c \to 0$.

*Remark 2.* The relativistic corrections to the Zeeman effect (Bethe and Salpeter, 1957), can be evaluated in terms of the variation corresponding to the gyromagnetic ratio, as follows: the magnetic term in the Dirac Hamiltonian is $-e\boldsymbol{\alpha}\cdot\mathbf{A}$, with $\mathbf{A}$ the vector potential. Since the magnetic field is constant and chosen in the $0Z$ direction

$$\mathbf{A} = \tfrac{1}{2}\mathbf{B}\times\mathbf{R} = \tfrac{1}{2}(-By, Bx, 0)$$

The correction to the energy in the first approximation is

$$\Delta\varepsilon = \frac{\int -e\psi^{+}\boldsymbol{\alpha}\cdot\mathbf{A}\psi\, d^3x}{\int \psi^{+}\psi\, d^3x} \tag{2.22}$$

and using (2.6) and (2.10) we get

$$\Delta\varepsilon = \tfrac{1}{2} g\mu_0 BM \tag{2.23}$$

where $g$ is given by (2.13) and (2.17) and $\mu_0$ is the Bohr magneton. Thus the relativistic corrections to the Zeeman effect are obtained in terms of the corrections to $g$.

## REFERENCES

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